ADDITIVE COMBINATORICS COURSE
PROBLEMS 2

(1) We saw in the course that $\max\{|A|,|B|\} \leqslant |AB| \leqslant |A||B|$ for any subsets A, B of a group G. Give examples to show that these bounds are tight.

(2) Given finite non-empty subsets A, B of a group G, the Ruzsa distance is

$$d(A,B) = \log\left(\frac{|AB^{-1}|}{\sqrt{|A||B|}}\right).$$

(a) Prove that $d(A,B)$ is non-negative and symmetric.

(b) Use the Ruzsa triangle inequality to prove that $d(A,C) \leqslant d(A,B) + d(B,C)$.

(c) Find an example illustrating that

$$d(A,B) = 0 \quad \not\Longrightarrow \quad A = B.$$

(3) Suppose that A is a finite symmetric subset of a group G containing the identity. Assume that $|A^3| \leqslant K|A|$.

(a) Prove that

$$\frac{|A^n|}{|A|} = \frac{\sqrt{|A^{n-2}||A^2|}}{|A|} \exp\left(d(A^{n-2}, A^2)\right).$$

(b) Using Ruzsa's triangle inequality prove that $|A^n| \leqslant K^{n-2}|A|$, for all $n \geqslant 3$.

(c) Let $G = \mathrm{SL}_2(\mathbb{F}_p)$ and let

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in G\right\}.$$

By considering the set $A = H \cup \{g\}$ show that A has small doubling, but not small tripling.

(4) Let A be a finite subset of a group G such that $|A^3| \leqslant K|A|$. Use the Ruzsa triangle inequality to prove that

$$|(A \cup A^{-1} \cup \{1_G\})^3| \ll K^3|A|.$$

(i.e. If A has small tripling then its "symmetrisation" has small tripling.)