ADDITIVE COMBINATORICS COURSE
PROBLEMS 1

(1) Prove the Fourier inversion theorem for finite abelian groups.

(2) Let $c_1, \dots, c_k \in \mathbb{Z}$. For a finite subset $A \subset \mathbb{Z}$, let $N_k(A)$ denote the number of $x_1, \dots, x_k \in A$ such that
$$c_1 x_1 + \cdots + c_k x_k = 0.$$
Express $N_k(A)$ in terms of the Fourier transform $\widehat{\mathbf{1}_A}$ of the characteristic function of A.

(3) By considering the set
$$\left\{ n \in [N] : n = \sum_{0 \leqslant i \leqslant k} a_i 3^i \text{ such that } a_i \in \{0,1\} \right\},$$
or otherwise, prove the lower bound $r_3(N) \gg N^{\log 2/\log 3}$.

(4) The purpose of this question is to prove that there exists a function $c : (0,1] \to (0,1]$ such that if $A \subset [N]$ has density at least α, then A contains at least $c(\alpha)N^2$ three term arithmetic progressions (including trivial ones with common difference 0).
   (a) Explain why it suffices to assume that N is sufficiently large in terms of α.
   (b) Use Roth's theorem to prove that that there exists $M \in \mathbb{N}$, depending only on α, such that any subset of $[M]$ of density at least $\alpha/2$ contains a non-trivial three term arithmetic progression.
   (c) By dividing $[N]$ into disjoint arithmetic progressions of common difference d and lengths between M and $2M$, for $d \leqslant N/10M$, show that A contains at least
$$\frac{\alpha N^2}{30M^2}$$
three term arithmetic progressions (that are not necessarily distinct).
   (d) Show that that no three term arithmetic progression is counted more than $2M$ times, and conclude that A contains at least $\alpha N^2/(60M^3)$ distinct three term arithmetic progressions.