TOPICS IN ANALYTIC NUMBER THEORY
EXERCISE SHEET 3

(1) (Tensor power trick of Deligne/Langlands.) Let H be a finite set. For each integer $q \geqslant 1$, let P_q be a finite set. For each $x \in P_q$, let $\alpha_x \colon H \to \mathbb{C}$ be a function.

(a) For each $x \in P_q$ and integer $k \geqslant 1$, prove that the power series

$$f_{x,k}(T) := \prod_{(h_1,\ldots,h_{2k}) \in H^{2k}} \frac{1}{1 - \alpha_x(h_1)\overline{\alpha}_x(h_2)\cdots\alpha_x(h_{2k-1})\overline{\alpha}_x(h_{2k})T} \in \mathbb{C}[[T]]$$

has non-negative real coefficients. (*Hint*: Write $1/(1-t) = \exp(\sum_{r\geqslant 1} t^r/r)$.)

(b) Let $\sum_{n\geqslant 1} a_k(n)n^{-s}$ be the series obtained by formally expanding the product

$$\prod_{q\geqslant 1}\prod_{x\in P_q} f_{x,k}(q^{-s}).$$

Suppose $\sum_{n\geqslant 1} a_k(n)n^{-2}$ converges for all k. Prove that $|\alpha_x(h)| \leqslant 1$ for all $x \in P_q$ and $h \in H$. (*Hint*: Use (a) and the fact that $f_{x,k}(|\alpha_x(h)|^{-2k}) = \infty$.)

(2) (Poisson summation in $(\mathbb{Z}/r\mathbb{Z}) \times \mathbb{R}$.) Let $r \geqslant 1$ be an integer. Let $w_r \colon \mathbb{Z}/r\mathbb{Z} \to \mathbb{C}$ be a function and let $w_\infty \colon \mathbb{R} \to \mathbb{C}$ be a Schwartz function. Prove that

$$\sum_{n\in\mathbb{Z}} w_r(n)w_\infty(n) = \sum_{m\in\mathbb{Z}} \hat{w}_r(m/r)\hat{w}_\infty(m/r),$$

where $\hat{w}_\infty(m/r) := \int_{\mathbb{R}} w_\infty(t)\exp(-2\pi imt/r)\,dt$ is the usual Fourier transform, and where

$$\hat{w}_r(m/r) := \frac{1}{r}\sum_{a\in\mathbb{Z}/r\mathbb{Z}} w_r(a)\exp(2\pi ima/r).$$

(*Hint*: Apply the usual Poisson summation formula to integers $n \equiv a \bmod r$.)

(3) Let $\mathbb{F}_q$ be a finite field of order $q = p^r$, where p is prime and $r \geqslant 1$ is an integer. Let

$$\mathrm{Tr}(x) := \sum_{j=0}^{r-1} x^{p^j} \in \mathbb{F}_p$$

for $x \in \mathbb{F}_q$. Let ζ_p be a primitive pth root of unity in $\mathbb{C}$. Prove that if $x \in \mathbb{F}_q$, then

$$\frac{1}{q}\sum_{a\in\mathbb{F}_q} \zeta_p^{\mathrm{Tr}(ax)} = \mathbf{1}_{x=0}.$$

(*Hint*: The map $\mathrm{Tr} \colon \mathbb{F}_q \to \mathbb{F}_p$ is $\mathbb{F}_p$-linear and surjective.)

(4) Let A be an $N \times N$ *Hankel matrix* with complex entries,[1] where $N \geqslant 2$. Let B be the $N \times (N-1)$ sub-matrix given by deleting the last column of A. Let C be the $(N-1)\times(N-1)$ sub-matrix of B given by deleting the last row of B. Prove that

$$\mathrm{rank}(B) \leqslant \max(\mathrm{rank}(A) - 1, N-2, \mathrm{rank}(C)).$$

(*Hint*: First handle the case where $\mathrm{rank}(A) = N$. If $\mathrm{rank}(A) \leqslant N-1$, then choose a non-zero column vector $X \in \mathbb{C}^N$ with $AX = 0$, and do casework on whether the last coordinate of X is zero or not.)

[1] This means the entries of A are given by $A_{ij} = f(i+j)$ for $1 \leqslant i, j \leqslant N$, for some function $f \colon \mathbb{Z} \to \mathbb{C}$. In other words, the skew diagonals of A are constant.