TOPICS IN ANALYTIC NUMBER THEORY
EXERCISE SHEET 2

(1) Let $k \geqslant 2$ be an integer. Compute the probability that there is no integer $m \geqslant 2$ such that m^k divides n.

(2) Compute the probability that two integers n_1 and n_2 are coprime. Hint: take the corresponding Bernoulli random variables on $\{1, \ldots, N\} \times \{1, \ldots, N\}$ and take their limit as $N \to \infty$.

(3) Let $\Omega(n)$ be the number of prime divisors of an integer $n \geqslant 1$, counted with multiplicity (e.g. $\Omega(12) = 3$). Let $\mathbb{P}_N$ denote the uniform probability measure on $\{1, \ldots, N\}$. Prove that

$$\mathbb{P}_N \left(\Omega(n) - \omega(n) \geqslant (\log \log N)^{1/4} \right) \leqslant (\log \log N)^{-1/4},$$

and deduce, assuming Erdös–Kac Theorem for $\omega(n)$, that the random variables

$$X_N : \{1, \ldots, N\} \to \mathbb{R} \qquad n \mapsto \frac{\Omega(n) - \log \log N}{\sqrt{\log \log N}}$$

also converges in law to Normal$(0, 1)$ as $N \to \infty$.

(4) For any integer $N \geqslant 1$, let $m(N)$ denote the set of integers that occur in the multiplication table for integers $1 \leqslant n \leqslant N$:

$$m(N) = \{ab : a, b \in \mathbb{Z},\ 1 \leqslant a \leqslant N,\ 1 \leqslant b \leqslant N\} \subset \{1, \ldots N^2\}.$$

Prove that

$$\frac{\# m(N)}{N^2} \to 0$$

as $N \to \infty$. You may assume Erdös–Kac Theorem for $\Omega(n)$ (as obtained in the previous question).