TOPICS IN ANALYTIC NUMBER THEORY
EXERCISE SHEET 1

(1) Prove that
$$N_{\mathbb{P}^n,H}(B) \sim \frac{2^n}{\zeta(n+1)} B^{n+1},$$
as $B \to \infty$.

(2) Let $\nu(B)$ denote the number of $(a,b,c,d) \in \mathbb{N}^4$ such that $a^3+b^3=c^3+d^3$ and $a,b,c,d \leqslant B$. Show that $\nu(B) = O_\varepsilon(B^{2+\varepsilon})$, for any $\varepsilon > 0$. You may assume the standard estimate for the divisor function.

(3) Let $e_q(\cdot) = \exp(\frac{2\pi i}{q}\cdot)$ and let
$$S_q = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \sum_{\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^n} e_q\left(aF(\mathbf{x})\right),$$
where $F \in \mathbb{Z}[x_1, \ldots, x_n]$ is a homogeneous polynomial. Prove that S_q is a multiplicative function of q.

(4) Let S be the set of unramified rational primes p that split in the cubic number field $\mathbb{Q}(2^{1/3})$. Use the Chebotarev density theorem to check the convergence of the Euler product
$$\prod_{\substack{p \equiv 1 \bmod 3 \\ p \in S}} \left(1 - \frac{1}{p}\right)\left(1 + \frac{6}{p} - \frac{6}{p^2}\right).$$
Similarly, check the convergence of
$$\prod_{\substack{p \equiv 1 \bmod 3 \\ p \notin S}} \left(1 - \frac{1}{p}\right)\left(1 - \frac{3}{p} + \frac{3}{p^2}\right).$$

(5) Let p be a prime and let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ be vectors that are both coprime to p, but with $p^2 \mid \mathbf{a}.\mathbf{b}$. Let M denote the set of vectors $\mathbf{x} \in \mathbb{Z}^n$ such that $p^2 \mid \mathbf{a}.\mathbf{x}$ and $\mathbf{x} \equiv \lambda \mathbf{b} \bmod p$, for some $\lambda \in \mathbb{Z}$.
  (a) Prove that M is a lattice of rank n.
  (b) Prove that $\det \mathsf{M} = p^n$.